Maintaining individuality through scaling with equitable transformation: Applications to tundra plant phenology and climate

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What is an Equitable system?

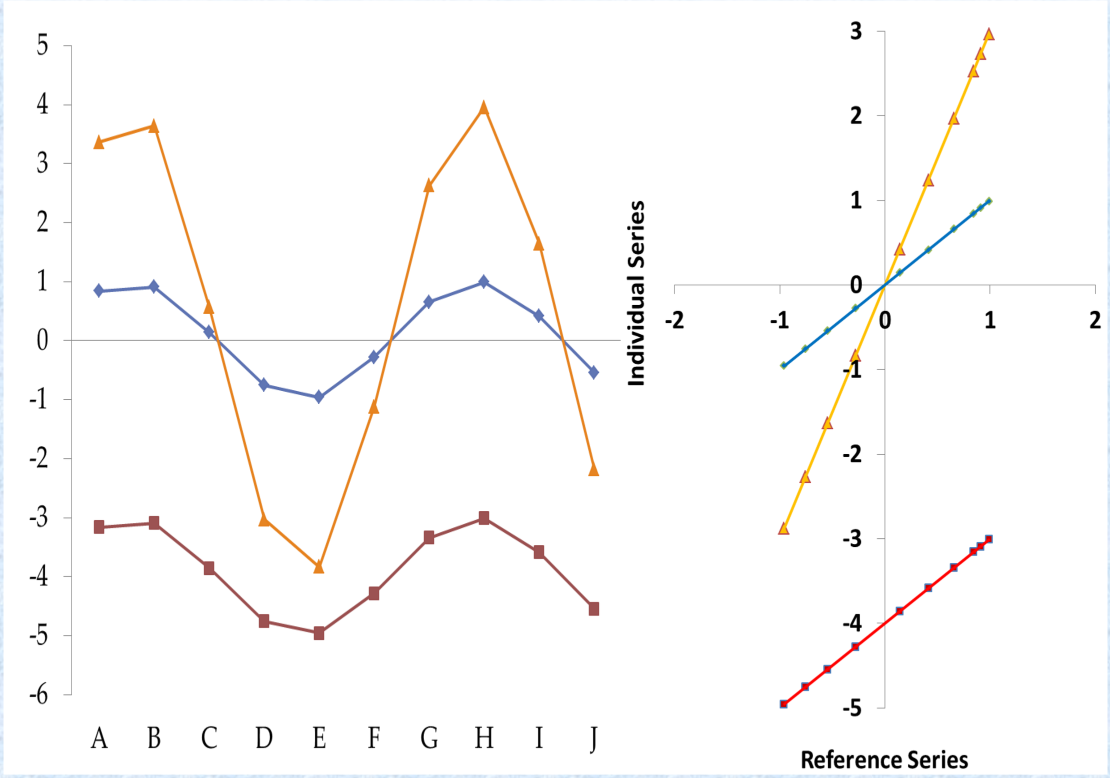


Figure 1: (LEFT) Three sequences (yellow, blue, red) (such as temperature at three spatial locations) as a function of arbitrary attributes A-J (such as times the temperature was measured at the three locations). **(RIGHT)** The three sequences plotted as a function of a reference sequence, chosen in this case to be the blue sequence from the left graph.

The three sequences shown above have a linear relationship with the arbitrarily chosen reference sequence (blue line, Figure 1). This is a simple example of an equitable system

When a data set L of NxM points has a perfectly equitable form, the N time sequences of M points will correlate perfectly with the reference sequence, and any sequence can be chosen as this reference sequence (x=r). It is a product of a spatial function f(x) and a temporal function g(t) plus a second spatial function u(x):

$$L(x,t) = f(x)\,g(t) + u(x).$$

When pairs of these time sequences are correlated, the resulting slope matrix is an "equitable" matrix. The two specific sequences (at locations z and y) can then be written as:

$$L_z(t) = f_z\,g(t) + u_z \text{ and } L_y(t) = f_y\,g(t) + u_y.$$

And are linearly related by:

$$L_y(t) = a_{yz}\,L_z(t) + b_{yz}.$$

This equitable matrix of slopes has particular properties that are:

$$a_{zz} = 1,\ a_{zy} = 1/a_{yz},\ \text{ and } a_{xy} = a_{xz}a_{zy}$$

for all indices x, y and z.

With the exclusion of the closure property, the properties of the elements of the equitable slope matrix define a mathematical group (Eves, 2012; Parker, 1965b). The matrix of intercepts also has special properties:

$$b_{yx} = b_{yz} - a_{yx}b_{xz},\quad a_{yx}b_{xz} = -a_{yz}b_{zx}\quad \text{and } a_{yx}b_{xy} + b_{yx} = 0$$

The transform converts values of the data at spatial locations z to values at x via the equitable transformation:

$$I(x,t) \sim T[I(z,t)] = AI + B$$

where A and B are the slope and intercept matrices described above.

Dealing with missing data

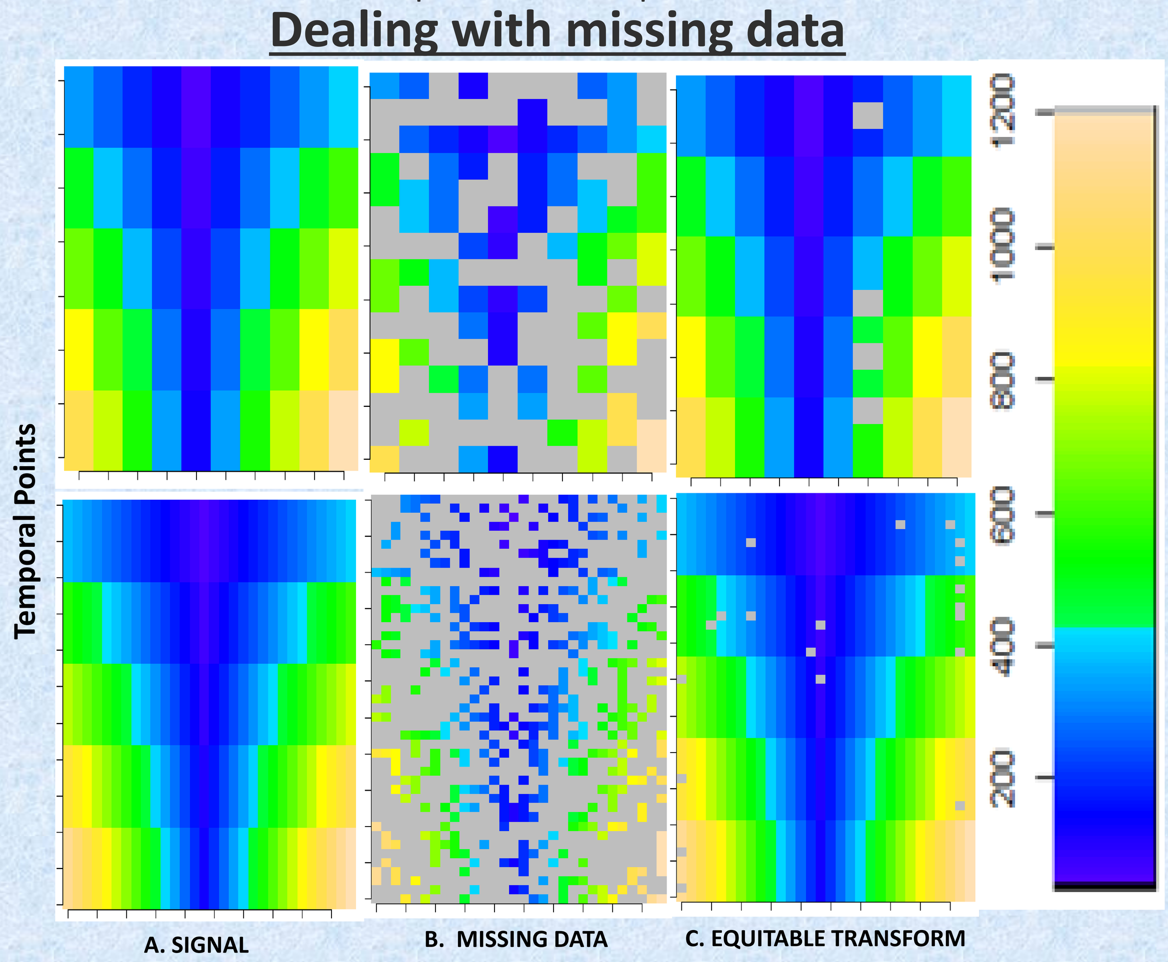


Figure 2: A colour coded example data set consisting of a step function in time (t) and an absolute value function in space (x). The signal (left panel), the signal with missing information (middle panel) and the recovered signal from an equitable transform (right panel) are shown. **Top row:** 15 temporal samples each at 10 spatial locations are taken at equal intervals . Middle top panel has 47% randomly missing data. **Bottom row:** The data set resolution increased by a factor of 3 (45 time samples x 30 spatial samples). The allowed fraction of randomly missing data (bottom middle) that was recoverable in the bottom right panel increased to 70%.

Recovering signal embedded in noise

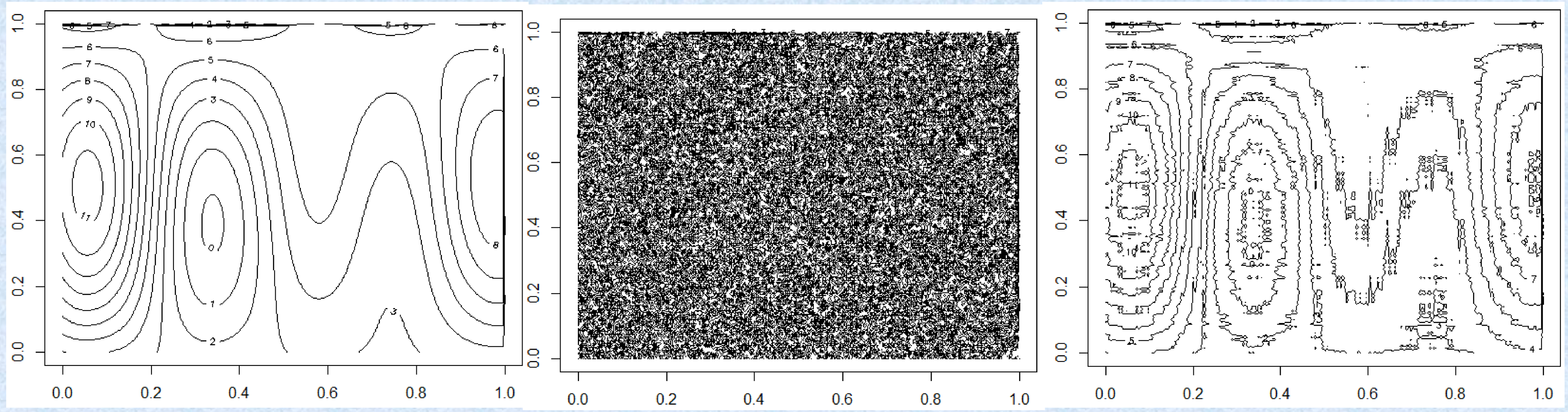


Figure 3: Contour maps of a separable signal **(LEFT),** the signal plus added Gaussian noise (σ_N=0.5σ_S) **(MIDDLE),** and the equitable transform based on the noisy data set **(RIGHT).** The separable signal shown here is composed of 150 spatial points equally spaced at 7.2 units up to 360. The 225 temporal points extended to 360 time units in increments of 1.6.

Addressing regression dilution bias

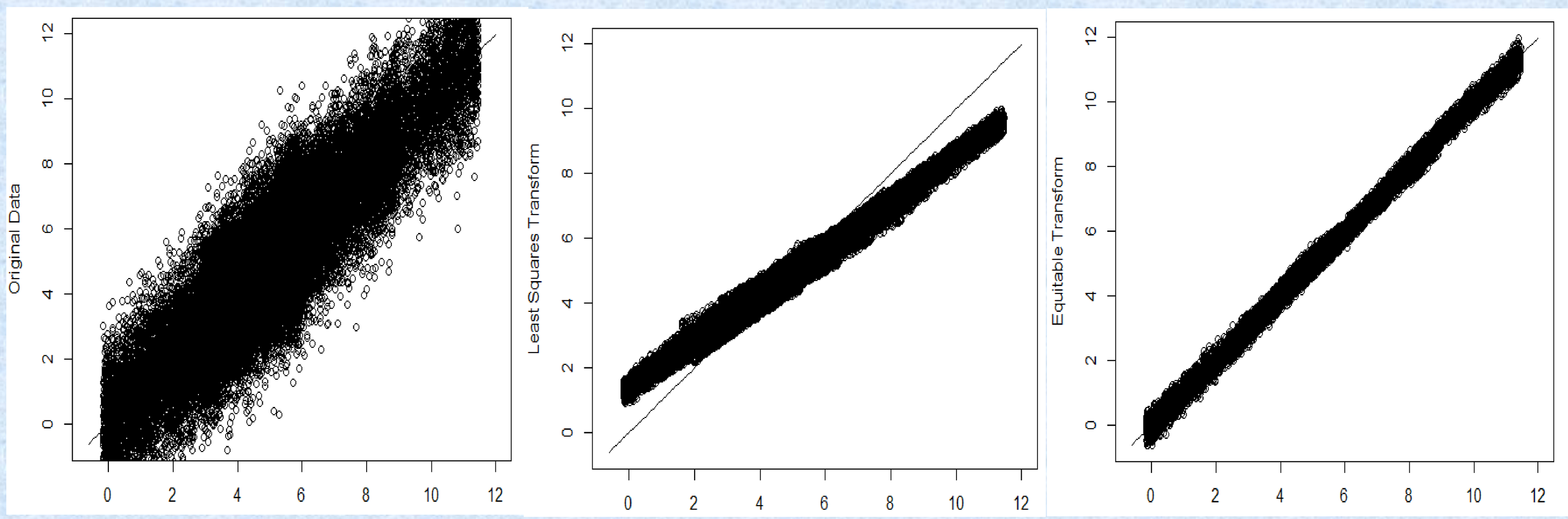


Figure 4: The signal + noise ($\sigma_N = 0.5\sigma_S$) **(left panel)**, the least squares transform **(middle panel)** and the equitable transform (**right panel**) versus the separable signal. Y=X is plotted for context.

Equitable High Arctic Temperatures

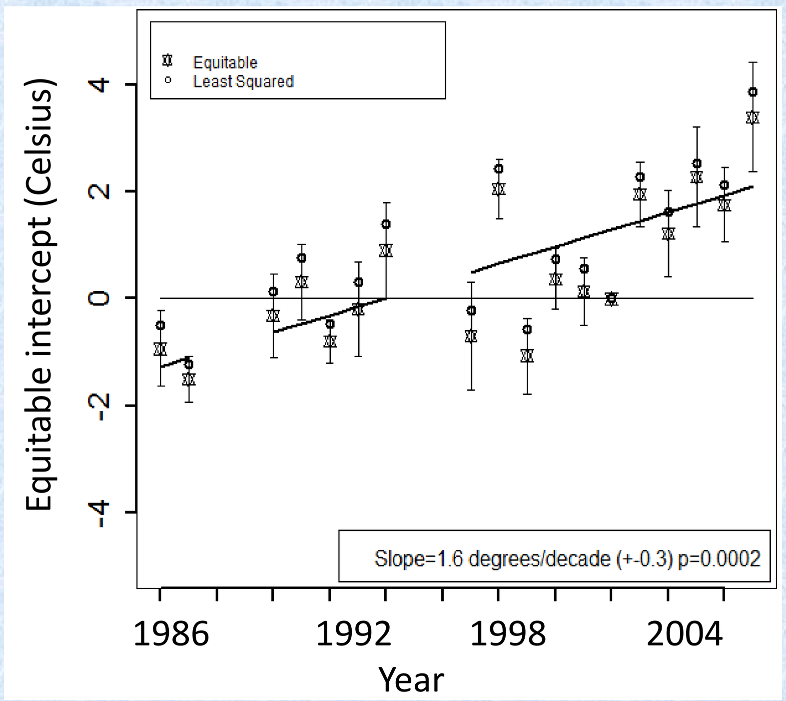


Figure 5: The long term trend of temperature relative to reference year (2002) shows an increase of 1.6±0.3 °C/decade, n=18, p-value =0.0002. This is a plot of the temperatures from 1986-2006 on the day of the year when temperature was -20°C in 2002. Error bars represent the standard deviation of the estimate of the equitable intercept formed from all years' observations. Larger per decade increases in temperature occur in during the winter than during the summer.

Tundra plant phenology as an equitable system

The equitable technique was applied to long term phenology data of *Dryas integrifolia* recorded at Alexandra Fiord, Ellesmere Island. Phenological cycles appear to be reasonably approximated using day number being a function of both individual plant and phenology event.

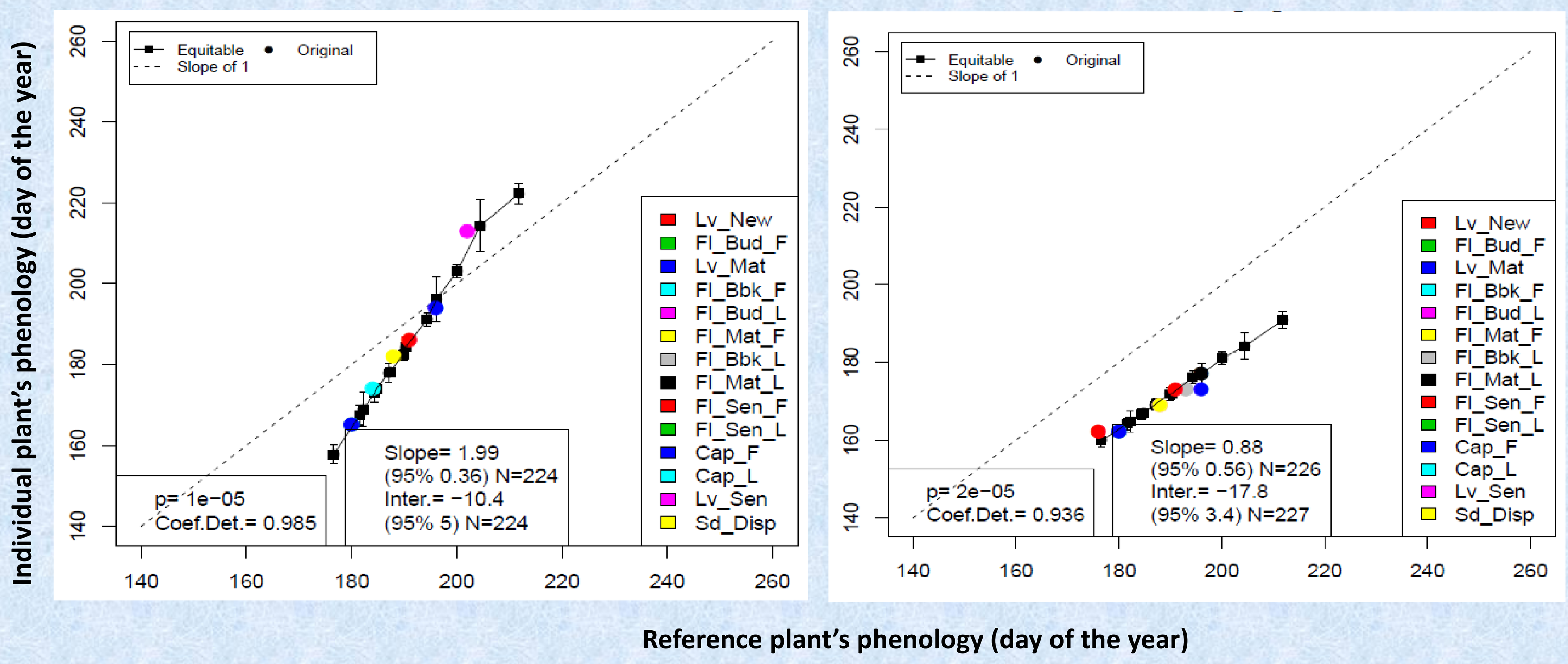


Figure 6: Phenology of two *D. integrifolia* plants from a dry-mesic site at Alexandra Fiord plotted versus a 1996 reference plant. The coloured points are the raw data and the solid squares with error bars correspond with the equitable model. The error bars represent the standard deviation of the model value based on all other plants generating that value. The dotted line shows the reference plant plotted against itself. Phenological stages are colored as in the legend.

The slope of the graph above describes the rate a plant progresses through the phenology stages. Plants with larger slopes indicate slower progression through phenology stages compared to the reference plant selected. Figure 7 illustrates the long term variation of the slope (relative to the average profile) from the equitable system. Plants appear to be proceeding more rapidly through their stages in the later years.

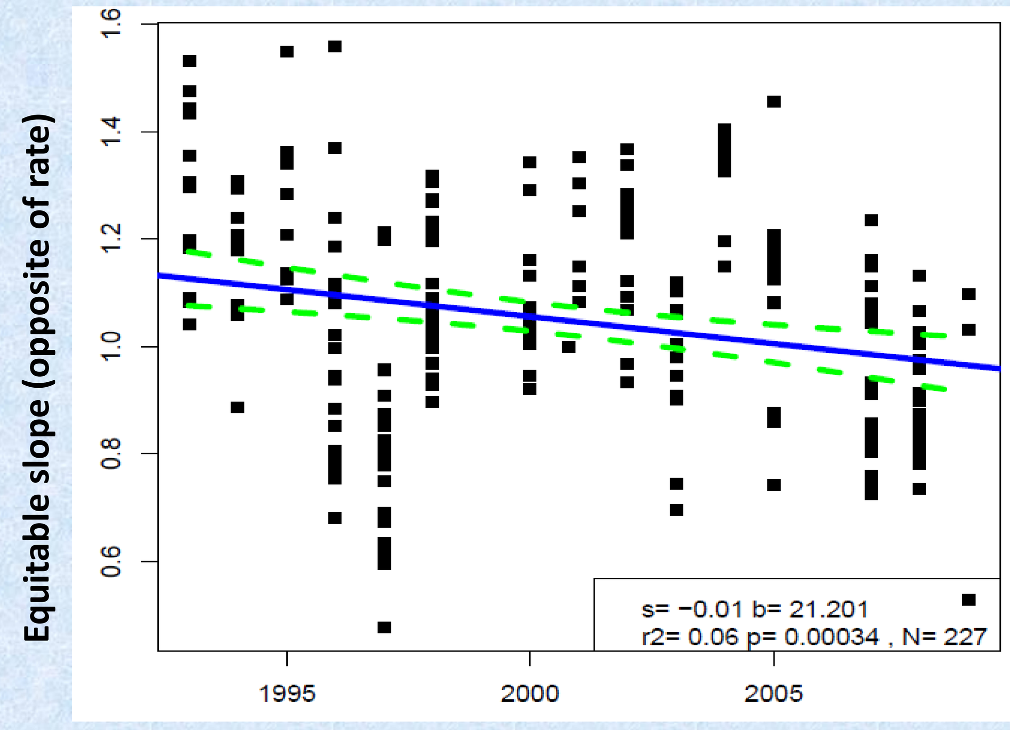


Figure 7: Equitable slopes of the *D. integrifolia* phenology system (opposite to the phenology rate) plotted against the year (1993 - 2009). The blue line is the regression fit and the green lines represent 95% confidence intervals. The equitable system was determined from 227 individual *D. integrifolia* plants from a dry-mesic site at Alexandra Fiord between 1993 and 2009.

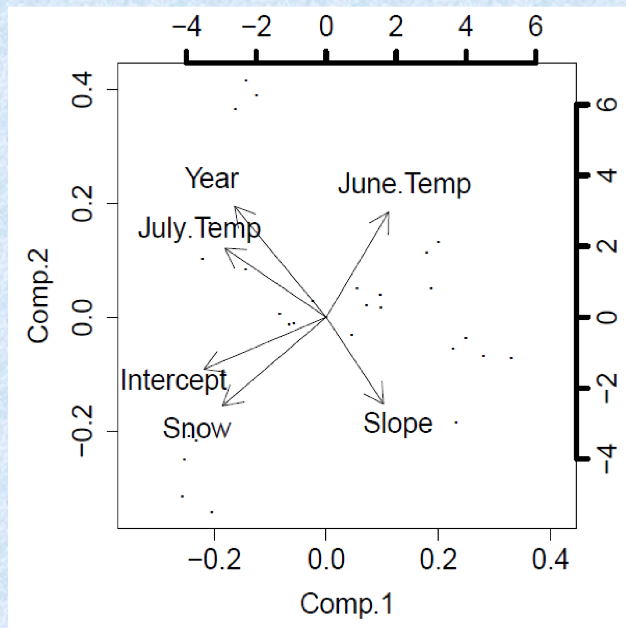
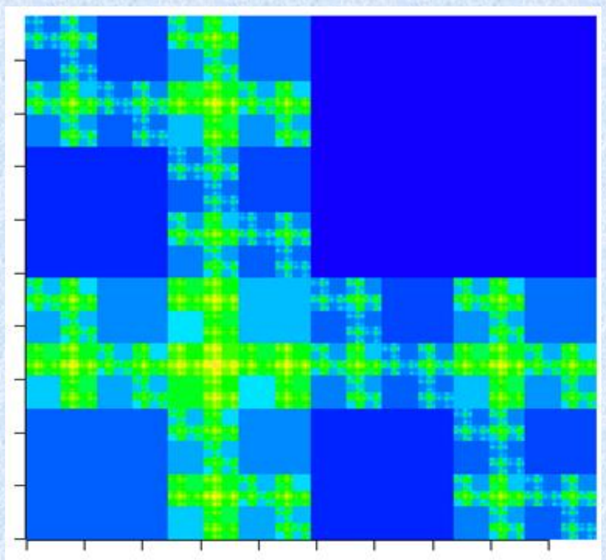


Figure 8: Principal component analysis of *D. integrifolia* plants **from one plot at the dry-mesic site between 1993 and 2009.** Factors shown include equitable slope and intercept, average June (June.Temp) and July (July.Temp) temperature, snowmelt date (Snow) and Year of observation. Increasing July temperatures are associated with decreasing slope in recent years.

If replications and equitable–like behavior occur at different scales, then multiple levels of equitable systems can be combined to form large complex non-equitable systems. Fractals such as the Sierpinski triangle are simple multi-level equitable systems. For the Sierpinski triangle each level is only 2 points in space and 2 points in time (Figure 9).

Figure 9: Fractal-like data set with 8 levels of simple equitable systems each of 2 points in space and 2 points in time. Rows and columns are labelled with the row (time) and column (space) coordinate of each of the 8 levels. Equitable systems allow scale size and grain size of levels to be found by correlations.



CONCLUSIONS

An equitable transform technique has been outlined that retains individuality of sequences by describing each sequence through the scaling and shifting of a general underlying function. Application of this equitable transform to long term temperature data on Ellesmere Island showed warming occurring at rates of 1.8 ±0.4 °C/decade at -27°C, 1.6±0.3 °C/decade at -20°C, 1.5±0.28 °C/decade at -14.7°C and 1.1±0.2 °C/decade at 0°C. A "phenological rate", using equitable systems applied to phenology, is found to decrease from 1993 to 2009 in association with increasing temperatures. This indicates that plants are progressing through their stages more rapidly with a warming climate.

REFERENCES

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Parker, F.D. (1965). When is a loop a group. American Mathematical Monthly *72*, 765–766.

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